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TECHNICAL REPORT

Office of Naval Research Contract No. N00014-86-K0029

M/G/1 WITH EXCEPTIONAL SERVICE AND ARRIVAL RATE

by

Martin Krakowski

Report No. GMU/22472/104
October 31, 1988

Department of Operations Research and Applied Statistics
School of Information Technology and Engineering
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M/G/1 with Exceptional Service and Arrival Rat

by

Martin Krakowski

Abstract

The model M/G/1 is modified by (a) providing the pioneer customer, i.e. the customer who terminates an idle period (and initiates a busy period) with exceptional service; and (b) by allowing an exceptional arrival rate during the idle period. The regimen is FCFS and the server idles only when customers are absent. We derive omni-equations for the delay, for the backlog, and for the queue size as found by real or virtual arrivals. We relate these processes to the regular M/G/1 as convolutions of the delay in M/G/1 with modifying variables in the model treated. The queue size is derived from the delay by applying the Poisson operator.

This paper extends the results of P.D Welch (1964) who allowed exceptional service but not exceptional arrival rate, and who did not discuss composition relations. His solutions used Laplace transforms and generating functions. It is believed ~~We believe~~ that the current treatment is simpler and more suitable for further generalizations. Introducing the exceptional arrival rate forces us to distinguish between the perception of the observer and of the customer source.

Notation

$\sigma=$ arrival rate when server idles

λ = arrival rate when server serves

y = service time for a "pioneer" (customer who initiates a busy period)

x= service time for non-pioneers, or "followers"

$\hat{z}$=residual service time of z or, briefly, the residue of z

$\rho = \lambda Ex = \lambda\mu \qquad \rho_0 = \sigma\bar{y}$

B=backlog or unfinished work

w = overall delay as perceived by the customer source

w_* = positive delay, i.e. delay of a follower; a pioneer's delay = 0

u= clearance time of the system (=virtual waiting time in our model)

u_* = clearance time of the system, provided the server works

n = queue size at a random instant (or viewed continuously)

n_*=queue size at a random instant provided the server works

q_d=queue size left behind by a departing customer who enters the service station

q_a=queue size found by an arriving customer

q_{*d} = queue size left behind by a departing follower who enters the queue

q_{*a}=queue size found by a follower (i.e.found by a customer while server works

N=system size as seen by a continuous (or Poissonian) observer

Q=system size found by a newcomer (or left behind by a served customer)

Q_0=fraction of newcomers who find the server idle

Q_*=fraction of newcomers who find the server busy

$P_0=\Pr(N=0) \qquad P_* = \Pr(N>0) = \Pr(\text{server serves}) = 1-P_0$

A *free copy* of a random variable z is a random variable having the same distribution as z and independent of any other variables within the same argument. Different free copies of the same variable are usually designated by the same symbol if they occur in different arguments; this causes no confusion.

Omni-Transform

Definition 1 The omni-transform of the random variable A is the expectation of an arbitrary function of A: $E\psi(A)$. The arbitrariness of ψ is limited only by the requirement that $E\psi(A)$ and $\psi'($ $)$ exist. When using this definition we usually apply the so-called omni-convention (see below).

Definition 2 The omni-transform of the random variable A is an arbitrary functional of A.

It can be shown that the two definitions are essentially equivalent since under wide enough conditions a functional of A can be represented in the form $E\psi(A)$ for some ψ. Among the advantages of Definition 2 is that we need not know the function whose expectation is the functional of interest and we can thus easier specialize $\psi(A)$. Thus, if we need $\Pr(A \leq t)$ we need not write $\psi(A) = EH(t-A)$ where $H(t-A)=1$ if $t \leq$ A and $=0$ otherwise; we simply write $\psi(A) = \Pr(A \leq t)$. Another advantage of Definition 2 is that no need arises for the omni-convention (see below). But perhaps Definition 1 is somewhat simpler when balancing $\psi(A)$. Since the omni-equations are typographically identical for both definitions (when the omni-convention is used with Definition 1) the reader may choose either.

Omni-Convention Omni-equations are easily told by sight. Hence $E\psi(A)$ can be replaced in print by $\psi(A)$ and the operator E is retained mentally; this is the omni-convention. Definition 2 has no need for the omni-convention. Omni-equations look the same under both definitions if the omni-convention goes with Definition 1. One can even jockey between the two definitions for didactic or esthetic reasons.

Section 1. The Balance of $\psi(B)$ in G/G/1 with Exceptional Service

It is of interest to see how far we can get within a modified G/G/1, in which the pioneer customer receives exceptional service, when we apply the method of omni-equations. (The generality of the arrival stream includes the possibility of exceptional rate for the pioneers.) Once we reach a barrier to further progress we will see clearer why that barrier may be lifted by a fundamental property of a steady Poissonian source, namely that true arrivals see the same picture, stochastically, as does a continuous observer or a Poissonian observer.

In our model the backlog B manifests itself to new arrivals as their *delay* w: we can say that "w = B conditioned upon a customer just to arrive" and it manifests itself to a continuous observer as the *virtual delay* u (or *clearance time in our model*). *An observer who samples the system at instants generated by an independent Poisson source of steady intensity is equivalent to a continuous observer. Clear intuitively that Poissonian sampling should be unbiased, this was also treated analytically* (*Wolff* 1982). Such an observer will be tersely referred to as a random observer and his observations as random observations. We can say that "u = B conditioned upon the observation instant being random." The process B is thus a superposition of the point process w and the continuous (almost everywhere) process u.

Under steady-state conditions B is a balanced process, which means that $E(dB) = 0$ during a random dt. But $\psi(B)$, an arbitrary function of B, is also balanced since we also have $d\psi(B) = 0$ over any random time interrval dt.

We shall now derive the balance equation for $\psi(B)$. *Balancing an arbitrary function of a process, rather than the process itself, is the essence of the omni- method.* The expected changes in $\psi(B)$ derive from two causes:

(a) While the server works the backlog B is being worked off continuously at the rate

$du = du_* = -dt$; and $du = 0$ while the server idles; thus the expected change in $\psi(B)$ due to the work-off is, using the omni-convention (i.e. by mentally taking the expectation of each side of the equation):

$$d\psi(B) = -dtP_*\psi'(u_*) \tag{1.1}$$

where $P_* =$ Pr(server works) = fraction of time as seen by a continuous observer. Of course, $P_0 + P_1 = 1$.

(b) New arrivals cause upward jumps in B. A pioneer causes a jump from 0 to y, and a follower causes a jump from w_* to $w_* + x$. Thus, $Ed\psi(B)$ caused by new arrivals during dt is

$$d\psi(B) = dt\,\Lambda\,Q_0[\psi(y) - \psi(0)] + dt\,\Lambda\,Q_*[\psi(w_*+x) - \psi(w_*)] \tag{1.2}$$

where Λ=global frequency of arrivals; Q_0=fraction of customers who find the server idle; and Q_*=fraction of customers who find the server busy. Note that the Q-weights refer to the experience of customers, while the P-weights refer to the experience of an observer. In a model with a steady Poissonian source of customers a continuous (or random) observer sees, statistically, what the arrivals see so that then $\psi(u) = \psi(w)$. But this is not generally the case for a non-Poissonian source (even for a piecewise Poissonian source as in our model) where customers may find different probabilities than do observers; thus $\psi(u) \neq \psi(w)$.

From (1.1) and (1.2) we get the balance equation for the backlog in G/G/1:

$$P_*\psi'(u_*) = \Lambda Q_0[\psi(y) - \psi(0)] + \Lambda Q_*[\psi(w_*+x_*) - \psi(w_*)] \tag{1.3}$$

Equation (1.3) can be integrated, i.e. brought into a form free of derivatives. It is easy to show (cf. Krakowski September 1984 and 1985) that

$$\psi(z) - \psi(0) = \bar{z}\,\psi'(\hat{z}) \tag{1.4}$$

for any positive random variable z which is interpreted as a renewal process; $\hat{z}$ is the residual time of z, or simply its residue. Equation (1.4) plays a key role in the treatment of queues when integrating omni-equations. *In fact, equation* (1.4) ***helps to exploit the view of a single-server queue as a renewal station modified by gaps in service.*** It follows from (1.3) and (1.4) that

$$P_* \psi'(u_*) = \Lambda Q_0 \bar{y} \psi'(\hat{y}) + \Lambda Q_* \bar{x} \psi'(w_* + \hat{x}) \tag{1.5}$$

Since $\psi'(\ )$ is also a general function of its argument we can replace in (1.5) each ψ' by ψ — we call this procedure typographical integration — thus obtaining for G/G/1

$$\psi(u_*) = \Lambda Q_0 \bar{y} \psi(\hat{y}) + \Lambda Q_* \bar{x} \psi(w_* + \hat{x}) \tag{1.6}$$

From (1.6) we get, upon setting $\psi(A)=1$, $\Lambda(Q_0\bar{y}+Q_*\bar{x})=1$ which along with $Q_0+Q_*=1$ yields Q_0 and Q_* in terms of Λ and $\bar{y}$ and $\bar{x}$.

Equation (1.6) can be also read in the following manner: The random variable u_* is a mixture of $\hat{y}$ and $w_*+\hat{x}$ with weights $\Lambda Q_0\bar{y}$ and $\Lambda Q_*\bar{x}$. Note that w_* and $\hat{x}$ are independent, as follows from the derivation of (1.6); unless otherwise indicated, each variable in each argument of an omni-equation is a free copy of its generic prototype: it is distributed like this prototype and is independent of any other variable within the argument.

The unknown processes in (1.6) are u_* and w_*. The variables y and x, and hence their residues $\hat{y}$ and $\hat{x}$, are known. It appears that the theory of the delay in G/G/1 and in some of its variants reduces to relating w_* to u_*. In order to solve (1.6) for both w_* and u_* we need another equation in these variables, wishfully an omni-equation. However, there is no assurance that such an omni-equation exists for all or most models with general input. If it does exist its form is likely to be complex. (No general equation relates the Laplace Transforms of u_* and w_*, which equation might then be transposed into an equivalent omni-equation.) A problem of applied and methodological interest is, for what models can w_* and u_* be related by means of omni-equations alone?

When the arrivals are Poissonian, at least during the busy period, such relations between u_* and w_* do exist and can be simple. From these, in turn, relations between u and w can be derived.

This is the subject of Section 2.

Section 2. Balancing $\psi(B)$ in M/G/1 with Exceptional Service and Arrival Rates

Like for G/G/1 we find it useful to start with the balance of $\psi(B)$ as a stepping stone towards the analysis of the delay w. Moreover, the backlog is of economic interest when some costs, e.g. inventory costs, depend on B.

Let f_{01}=frequency of transitions of system size from N=0 to N=1. Clearly, with Λ=global arrival rate, and with σ and λ being the Poissonian intensities during the server's idle and busy periods, we have

$$\Lambda = \sigma P_0 + \lambda P_* = \Lambda Q_0 + \Lambda Q_* ; \quad P_0 + P_* = 1 \text{ and } Q_0 + Q_* = 1 \tag{2.1a}$$

$$f_{01} = \sigma P_0 = \Lambda Q_0 \tag{2.1b}$$

From (2.1a) and (2.1b) we get

$$\lambda P_* = \Lambda Q_* \tag{2.1c}$$

Note now that for our current variant of M/G/1 we have the key property

$$\psi(u_*) = \psi(w_*) \tag{2.2}$$

The continuous observer who sees u_*, is equivalent to a random observer of u_* who takes readings with frequency λ and who must see what true arrivals see. We have ψ(Poissonian u_*) = ψ(continuous u_*) in a self-explaining way.

From (1.6), (2.1.b), (2.1c) and (2.2) we get

$$P_* \psi(w_*) = \sigma \bar{y} P_0 \psi(\dot{x}_0) + \lambda \bar{x} P_* \psi(w_* + \dot{x}) \tag{2.3}$$

an omni-equation with w_* as its only unknkown random variable. Dividing each term in (2.3) by P_* and defining

$$\rho_0 = \sigma \bar{y} \quad \text{and} \quad \rho = \lambda \bar{x}$$

we obtain

$$\psi(w_*) = (\rho_0 P_0 / P_*)\psi(y) + \rho\psi(w_* + \hat{x}) \tag{2.4}$$

With $\psi(\ )=1$ in (2.4) we find that

$$\frac{\rho_0 P_0}{P_*} = 1-\rho \tag{2.5}$$

and

$$\boxed{\psi(w_*) = (1-\rho)\psi(\hat{y}) + \rho\psi(w_* + \hat{x})} \tag{2.6}$$

Note that the differential equation (1.3) includes in its arguments y and x, the service durations, whereas the integrated equation (2.6) includes in their stead their residues. This is a characteristic feature of integrating a differential omni-equation in the context of queueing analysis: Residual service times tend to go with integrated equations. This explains why it has been repeatedly observed that the formal structure of M/G/1 and its variants is simplified when expressed in terms of $\hat{x}$ rather than x.

From (2.5) and $P_0 + P_* = 1$ we obtain

$$P_0 = \frac{1-\rho}{1-\rho+\rho_0} \quad \text{and} \quad P_* = \frac{\rho}{1-\rho+\rho_0} \tag{2.7}$$

and from (2.7) and (2.1) we find Λ and Q_0 and Q_*.

We find the successive moments of w_* by setting $\psi(A) = A^k$ in (2.6) and get

$$Ew_* = Ey + \frac{\rho}{1-\rho} E\hat{x} \tag{2.8}$$

With $\psi(A) = e^{-sA}$ in (2.6) we find the Laplace Transform of w_*:

$$Ee^{-sw_*} = \frac{(1-\rho)Ee^{-s\hat{y}}}{1-\rho Ee^{-s\hat{x}}} \tag{2.9}$$

We have often found that the easiest way to derive the Laplace Transform of a random process is via an omni-equation. And we find an integral equation for the cumulative distribution function of w_* by setting $\psi(A) = \Pr(A \le t)$ in (2.6), thus obtaining the convolution equation

$$\Pr(w_* \leq t) = (1-\rho)\Pr(\hat{y} \leq t) + \rho \Pr(w_* + \hat{x} \leq t) \tag{2.10}$$

One can even find the queue size by specializing the function or functional $\psi(A)$; cf. Section 5.

The reader may have noticed already that (1.6) implies (2.6) when $\psi(u_*) = \psi(w_*)$ and $\Lambda Q_0 = \sigma P_0$ and $\Lambda Q_* = \lambda P_*$ but in order to develop the M/G/1 model independently of G/G/1 we started again with the basics.

Section 3. A Composition Theorem for $\psi(\mathbf{w}_*)$

The omni-equation for the delay w in a regular M/G/1 is, in integrated form (cf. Krakowski 1986),

$$\psi(\underset{R}{w}) = (1-\rho)\psi(0) + \rho\psi(\underset{R}{w}+\hat{x}) \tag{3.1}$$

The R beneath the w is a reminder that this w pertains to the regular M/G/1 .

<u>*Theorem*</u> *1 Suppose that certain positive random variables Z and g satisfy the omni-equation*

$$\psi(Z) = (1-\rho)\psi(g) + \rho\psi(Z+\hat{x}) \tag{3.2}$$

in which ρ and $\hat{x}$ are as in (3.1). *Then the following composition holds*

$$\psi(Z) = \psi(\underset{R}{w}+g) \tag{3.3}$$

which says that Z is distributed like the sum of the generic variables $\underset{R}{w}$ and g.

Proof Shifting (3.1) by g we obtain

$$\psi(\underset{R}{w}+g) = (1-\rho)\psi(g) + \rho\psi(\underset{R}{w}+\hat{x}+g) \tag{3.4}$$

From (3.2) and (3.4) we get

$$\psi(Z)-\rho\psi(Z+\hat{x}) = \psi(\underset{R}{w}+g) - \rho\psi(\underset{R}{w}+\hat{x}+g) \tag{3.5}$$

Defining

$$\phi(Z) = \psi(Z) - \rho\psi(Z+\hat{x}) \tag{3.6}$$

we g: from (3.5)

$$\phi(Z) = \phi(g+\underset{R}{w}) \tag{3.7}$$

which, ϕ being a general function, is equivalent to (3. 3) ; this completes the proof.

Among the models for which (3.3) holds is the multiple vacation model in whicvh $Z=w$ and $g =$ v, a vacation period. A more general problem than the simple composition in terms of $\underset{R}{w}$ is to express an unknown process, e.g. w or w_* or B or B_*, in terms of other already known processes, not perforce the delay in M/G/1 and not perforce simple convolutions.

From (3.2), (3.7) and (2.6) we have the composition

$$\boxed{\psi(\mathbf{w}_*) = \psi(\underset{R}{\mathbf{w}} + \hat{y})} \tag{3.8}$$

Equation (3.8) solves the problem of deriving the delay in our model in a simple and versatile way.

We are now ready to derive the omni-equations for the delay w and for the virtual delay u.

Section 4 The Delay in M/G/1 with Exceptional Service and Frequency of Arrivals

The general delay w is a mixture of two delays, pioneers' and followers', with weights Q_0 and Q_* as perceived by the source of customers. The delay w thus clearly satisfies the omni-identity

$$\psi(w) = Q_0\psi(0) + Q_*\psi(w_*) \tag{4.1}$$

Hence, taking account of (3.8), i.e. $\psi(w_*) = (\underset{R}{w} + \hat{y})$, we have

$$\boxed{\psi(w) = Q_0\psi(0) + Q_*\psi(\underset{R}{w} + \hat{y})} \tag{4.2}$$

which is a valid solution for w since all coefficients and arguments on the right-hand side of the equation are known; Q_0 and Q_* are given in (2.1).

The steady or Poissonian observer will write an omni-equation for the virtual delay u as a mixture of pioneers' and followers' components, namely

$$\psi(u) = P_0\psi(0) + P_*\psi(u_*) \tag{4.3}$$

and since in our model $\psi(w_*) = \psi(u_*)$ we can also write, aided by (3.8),

$$\boxed{\psi(u) = P_0\psi(0) + P_*\psi(\underset{R}{w} + \hat{y})} \tag{4.4}$$

Since in the regular M/G/1 $\psi(\underset{R}{u}) = \psi(\underset{R}{w})$ we can also write (4.4) as

$$\psi(u) = P_0\psi(0) + P_*\psi(\underset{R}{u} + \hat{y}) \tag{4.4a}$$

It is seen thus that the omni-equations for w and for u, i.e. equations (4.2) and (4.4), differ only in their weights for finding the server idle or busy.

If $\sigma = \lambda$ then $P_0 = Q_0$ and $P_* = Q_*$ and we have $\psi(u) = \psi(w)$, in addition to $\psi(u_*) = \psi(w_*)$, since the steady (or Poissonian) observer gets the same statistical picture as does the source of customers, both pioneers and followers. This is M/G/1 with exceptional service but uniform arrival intensity, dealt with by Welch.

In some other variants of M/G/1 the property that $\psi(u_*) = \psi(w_*)$, without $\psi(u) = \psi(w)$, has also provided the key to the treatment of the queue size and the delay under steady-state conditions. This was the case, for example, in the *quorum problem*, alias *Heyman's N-Policy* (Cf. Krakowski, July 1986 and November 1986.) Indeed, when analysing a variant of M/G/1 it is worth considering under what conditions observers and arrivals see the same statistical picture.

Section 5 Poisson Operator and Queue Size

Let us define the following function of the time interval z:

$\#_\beta(z)=$ number of events begot by a Poisson source of intensity β during z.

We refer to $\#_\beta$ as the Poisson operator of intensity β, acting on z. We assume that the Poisson source is independent of z and of any process entering the model unless said otherwise. Where no confusion threatens we may omit the subscript β and the qualifier "of intensity β," thus writing $\#z$ in place of $\#_\beta(z)$.

The distribution of $\#_\beta(z)$ is well known (cf. e.g. Gross and Harris 1985):

$$\Pr(z=j) = E\,\frac{e^{-\beta z}\,(\beta z)^j}{j!} \tag{5.1}$$

The Poisson operator has two important operational properties.

(1) $\#(a+b)=\#a+\#b$ if the intervals a and b are independent, in particular if they do not overlap. In omni-notation

$$\psi(\#(a+b))=\psi(\#a+\#b) \tag{5.2}$$

(2) If z is a mixture of the random intervals a and b with respective weights α and $1-\alpha$ then $\#z$ is a like mixture of $\#a$ and $\#b$. In omni-notation

$$\psi(z)=\alpha\psi(a)+(1-\alpha)\psi(b) \;\Rightarrow\; \psi(\#z)=\alpha\psi(\#a)+(1-\alpha)\psi(\#b) \tag{5.3}$$

It follows from the above two properties that each integral omni-equation with constant coeficients stays valid if each of its arguments is subjected to a Poisson operator of the same intensity. We can think of the Poisson operator as a random clock, a Poisson clock, which assigns a random discrete measure to any time interval by means of the probabilities (5.1).

The interpretation of the resulting omni-equation has to proceed most carefully. In all our applications the Poisson operator will have the intensity of the customer source for the entire process or for the busy period. This choice often leads to omni-equations with arguments which are free copies of important discrete processes, such as n or n_*.

Of course, in general $\psi(\#(z_1+z_2)) \neq \psi(\#z_1)+\psi(\#z_2)$, even if z_1 and z_2 do not overlap.

Consider now the omni-equation for the regular M/G/1, i.e. when $\sigma=\lambda$ and $\psi(y)=\psi(x)$, and by implication $Q_0=P_0$,

$$\text{M/G/1} \qquad \psi(w) = (1-\rho)\psi(0) + \rho\psi(w+\hat{x}) \tag{5.4}$$

Let now $\#=\#_\lambda$ and let $\psi(A)=\phi(\#A)$ for any argument A in (5.4).

Equation (5.4) becomes

$$\text{M/G/1} \qquad \phi(\#w) = (1-\rho)\phi(\#0) + \phi(\#(w+\hat{x})) \qquad \#=\#_\lambda \tag{5.5}$$

and since $\#(w+\hat{x}) = \#w+\#\hat{x}$ equation (5.5) becomes

$$\text{M/G/1} \qquad \phi(\#w) = (1-\rho)\phi(\#0) + \phi(\#w+\#\hat{x}) \qquad \#=\#_\lambda \tag{5.6}$$

It is clear that in M/G/1 $\#w$ has the same distribution as q_d, size of queue left behind by an entrant into the service station; hence $\psi(\#w)=\psi(q_d)$. The distribution of q_d is like that of n, the number of customers seen by a random observer and which in turn is distributed like q_a, the number of customers found by a newcomer. Thus (cf. Krakowski 1974)

$$\text{M/G/1} \qquad \psi(\#w) = \psi(q_d)=\psi(q_a)=\psi(n) \tag{5.7}$$

. Clearly, $\#0=0$ with probability 1. And finally, $\#\hat{x}$ = number of Poisson events during $\hat{x}$ is assumed known along with x.

We therefore find that (5.5) becomes

M/G/1 $\boxed{\psi(n)=(1-\rho)\psi(0) + \rho\psi(n+h)}$ where $h=\#\hat{x}$ (5.8)

The power of the Poisson operator to recycle omni-equations for the delay into omni-equations for the queue size is not widely known. In fact, it may be new.

Note It can be shown that (cf. Appendix) with $\#x = k$ and with $\#\hat{x}=h$ we have the omni-equation

$$\boxed{\psi(k)-\psi(0) = \rho[\psi(h+1)-\psi(h)]} \tag{5.9}$$

thus expressing the functionals of h in term of the functionals of k which have been more often dealt with in the literature. (Cf. e.g. the k-matrix in Gross and Harris.)

We now supplement Theorem 1 of Section 3 by its analogue for the queue-size.

<u>Theorem</u> <u>2</u> *Suppose that the non-negative and integer-valued random variables M and j satisfy the omni-equation*

$$\psi(M)=(1-\rho)\psi(j)+\rho\psi(M+h) \qquad h=\#\hat{x} \tag{5.10}$$

in which ρ and h are as in (5.8). *Then the following composition holds*

$$\psi(M)=\psi(\underset{R}{n}+j) \tag{5.11}$$

The proof is analogous to the proof of Theorem 1 and shall be omitted.

We are now ready to recycle (2.6), the omni-equation for the positive delay w_*, into an omni-equation for n_*, the queue size while the server works, in our model M/G/1 with exceptional service and exceptional arrival rate.

$$\psi(w_*)=(1-\rho)\psi(\hat{y})+\rho\psi(w_*+\hat{x}) \tag{2.6)=(5.10}$$

Applying the Poisson operator to each argument in (5.10) we obtain

$$\psi(\#w_*)=(1-\rho)\psi(\#\hat{y})+\rho\psi(\#w_*+\#\hat{x}) \tag{5.11}$$

Let now q_{*d}=size of queue left behind during a positive delay; $q_{*a}=\#$in queue found by a follower: $n_* = $ #in queue when server works.

Clearly, $\psi(\#w_*) = \psi(q_{*d})$; $\psi(q_{*d}) = \psi(q_{*a})$ follows, as can be shown, from the fact that transitions "$n=j \rightarrow n=j+1$" are as frequent as "$n=j+1 \rightarrow n=j$" in systems where all events are single with probability one (even for G/G/1 and more general systems); and $\psi(n_{*a}) = \psi(n_*)$ because the source of customers and a continuous observer see the same statistical picture in a queuing system with a steady Poisson source, and in our model the source is steady when the server works. (Cf. Krakowski 1973, Theorem B) Therefore

$$\psi(\#w_*) = \psi(q_{*d}) = \psi(q_{*a}) = \psi(n_*) \tag{5.12}$$

From (5.11) and (5.12) we obtain

$$\boxed{\psi(n_*) = (1-\rho)\psi(\#\hat{y}) + \psi(n_* + h)} \tag{5.13}$$

From Theorem 2 and (5.13) it follows that

$$\boxed{\psi(n_*) = \psi(\underset{R}{n} + \#\hat{y})} \tag{5.14}$$

We can also derive (5.14) if we apply the Poisson operator to each argument in (3.8):

$$\psi(w_*) = \psi(\underset{R}{w} + \hat{y}) \qquad (3.8)=(5.15)$$

without the use of Theorem 2.

We have clearly

$$\psi(q_d) = Q_0\psi(0) + Q_*\psi(q_{*d}) \tag{5.16}$$

and since $\psi(q_{*d}) = \psi(n_*)$, as stated in (5.12), it follows from (5.14) and (5.16) that

$$\boxed{\psi(q_d) = Q_0\psi(0) + Q_*\psi(\underset{R}{n} + \#\hat{y})} \tag{5.17}$$

Equation (5.17) applies to the arrival and departure streams from the point of view of the customer source. Recall that $\psi(q_d) = \psi(q_a)$. The steady observer notes that

$$\psi(n) = P_0 \psi(0) + P_* \psi(n_*) \tag{5.18}$$

and from (5.14) and (5.18) he gets

$$\boxed{\psi(n) = P_0 \; \psi(0) + P_* \psi(\underset{R}{n} + \#\hat{y})} \tag{5.19}$$

the omni-equation applicable to his experience. We note that equations (5.17) and (5.19) differ only in their weights with which an idling or a working server is found, as it ought to be.

Section 6: System Size and System Sojourn

At times we may be interested in the system size and the system sojourn in our model of M/G/1 modified by exceptional service and arrival rate as described above. We can easily write down omni-equations for these processes from the results for the queue size and the delay. Thus ,

$$\psi(Q)=Q_0\psi(0)+Q_*\psi(1+q_*)=Q_0\psi(0)+Q_*\psi(1+n_*) \tag{6.1}$$

where Q=system size found by a newcomer. Note that "system size found by a newcomer" and "system size left behind by a served customer" have the same distribution (Cf. Krakowski 1974).

The system size N perceived by the continuous observer is described by the omni-equation

$$\psi(N)=P_0\psi(0)+P_*\psi(1+n_*) \tag{6.2}$$

The sojourn time W in the system is described by the omni-equation

$$\psi(W)=Q_0\psi(y)+Q_*\psi(w_*+x) \tag{6.3}$$

The equations (6.1) and (6.2) and (6.3) are self-explaining as mixtures of idle and busy terms. Note that when $N>0$ then there is one customer in the service station and n_* customers in the queue: hence the argument $1+n_*$ in (6.2). A similar argument holds for (6.1).

Appendix: Relating $\#x$ to $\#\hat{x}$

Consider the omni-equation of the renewal process generated by the generic service time x (cf. Krakowski 1987):

$$\psi(x)-\psi(0)=\bar{x}\,\psi'(r) \qquad r=\hat{x} \tag{A.1}$$

or

$$\psi(x)-\psi(0)=\lim \frac{\bar{x}}{dt}[\psi(r+dt)-\psi(r)] \tag{A.2}$$

Applying the Poisson operator to each argument in (A.2) we have

$$\psi(\#x)-\psi(\#0)=\lim \frac{\bar{x}}{dt}[\psi(\#+\#t)-\psi(\#)] \tag{A.3}$$

Denoting $\#x=k$ and $\#r=\#\hat{x}=h$ we write (A.3) as

$$\psi(k)-\psi(0)=\lim \frac{\bar{x}}{dt}[\psi(h+\#dt)-\psi(h)] \tag{A.4}$$

It is clear that

$$\begin{aligned}\psi(h+\#dt)&=\Pr(\text{one Poisson event})\psi(h+1)+\Pr(\text{no Poisson event})\psi(h) + 0dt^2\\ &=dt\lambda\psi(h+1)+(1-dt\lambda)\psi(h) + 0dt^2\end{aligned}$$

and the right-hand side of (A.4) becomes

$$\rho[\psi(h+1)-\psi(h)] \ \text{ since } \ \lambda\bar{x}=\rho \tag{A.5}$$

From (A.4) and (A.5) we have

$$\boxed{\psi(k)-\psi(0)=\rho[\psi(h+1)-\psi(h)]} \qquad (5.9)=(A.6)$$

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